Math 4200
Wednesday October 14
2.4 Consequences of Cauchy's integral formula: The fundamental theorem of algebra; Morerra's theorem and uniform limits of analytic functions; mean value property for analytic and harmonic functions. We'll carefully finish the fundamental theorem of algebra proof first, from Monday. I copied part of that page into today's notes because I'd used up the writing space there.

Announcements:

Fundamental Theorem of Algebra Let

$$p(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$$

be a polynomial of degree n (scaled so that the coefficient of z^n is 1), with $a_j \in \mathbb{C}$. Then $p(z)$ factors into a product of linear factors,

$$p(z) = (z - z_1)(z - z_2) \cdots\cdots (z - z_n).$$

proof:

- It suffices to prove there exists a single linear factor when $n \geq 1$ since the general case then follows by induction.

- To show that $p(z)$ has a linear factor, it suffices to show that $p(z)$ has a root, $p(z_1) = 0$. This follows from the division algorithm and dividing $p(z)$ by $z - z_1$:

$$p(z) = (z - z_1)q_{n-1}(z) + R$$

where R is the remainder. So $p(z_1) = 0$ if and only if $(z - z_1)$ is a factor of $p(z)$.

Then the proof proceeds by contradiction: If $p(z)$ has no roots, then $\frac{1}{p(z)}$ is entire, and

$$\frac{1}{p(z)} = \frac{1}{z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0} = \frac{1}{z^n} \frac{1}{\left(1 + \frac{a_{n-1}}{z} + \ldots + \frac{a_1}{z^{n-1}} + \frac{a_0}{z^n}\right)}.$$

We can show that $\frac{1}{p(z)}$ must be bounded, so by Liouville's Theorem it must be constant. This is a contradiction!

Estimates: We used a first derivative estimate via C.I.F. to prove Liouville's Theorem. Estimates for all derivatives are sometimes useful, and the most useful case is for the derivative estimate in the center of a disk.

Let $f: A \to \mathbb{C}$ analytic, (A open as always ... our running assumption on domains is that they are open connected sets, not necessarily simply connected though.) Let the closed disk $\bar{D}(z_0; R) \subseteq A$. Let γ be the circle of radius R, traversed once counterclocwise, so $I(\gamma; z_0) = 1$. Then we have the C.I.F and C.I.F. for derivatives,

$$f'(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^2}\, d\zeta$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta \, .$$

Let M be the maximum of $|f|$ on $\bar{D}(z_0; R)$, so also a bound for $|f|$ on γ. Then

$$\left|f'(z_0)\right| \le \frac{1}{2\pi}\int_\gamma \frac{|f(\zeta)|}{|\zeta - z_0|^2}\,|d\zeta| \le \frac{M}{2\pi}\frac{1}{R^2}2\pi \mathrm{R} = \frac{\mathrm{M}}{R}\, .$$

$$\left|f^{(n)}(z_0)\right| \le \frac{n!}{2\pi}\frac{M}{R^{n+1}}2\pi \mathrm{R} = \frac{n!\, M}{R^n}\, .$$

We used the first derivative estimate to prove Liouville's Theorem on Monday. You have the opportunity to use the higher order derivative estimates in your homework this week.

We'll use the following result, and especially its corollary, at key points of Chapter 3.
It's a converse to the theorem that the Rectangle lemma holds when $f(z)$ is analytic:

<u>Morera's Theorem</u> Let $f: A \rightarrow \mathbb{C}$ be continuous, and suppose the rectangle lemma holds, i.e.

$$\forall\ R = \{z = x + i\,y \mid a \leq x \leq b,\ c \leq y \leq d\} \subseteq A,$$
$$\int_{\delta R} f(z)\ dz = 0.$$

Then f is actually analytic on A.

proof:

Corollary Let $\{f_n\}: A \to \mathbb{C}$ analytic. Suppose $\{f_n\} \to f$ uniformly on A. Then $f: A \to \mathbb{C}$ is also analytic. (Contrast this with the analogous false theorem for differentiable functions on subdomains of $\mathbb{R}$).

proof: Can you check these pieces, and combine them into a proof?

(i) f is continuous, because uniform limits of continuous functions are continuous. (3210-3220?)
(ii) If $\{f_n\} \to f$ uniformly on A and if the rectangle lemma holds for each f_n (which it does, because each f_n is analytic), then the rectangle lemma holds for f.

One of the most-studied analytic functions is the *Riemann -Zeta function*. It is customary to write the complex variable as s in this case, rather than z. And for $\mathrm{Re}(s) > 1$, the Zeta function $\zeta(s)$ is defined by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$$

where for $s = x + i\,y$, each term

$$n^{-s} = \mathrm{e}^{-s\log(n)} = \mathrm{e}^{-(x+i\,y)\ln(n)} = n^{-x}\mathrm{e}^{-i\,y\ln(n)}$$

is analytic in s. Note that for $x > 1$, the sum of moduli

$$\sum_{n=1}^{\infty}\left|\frac{1}{n^s}\right| = \sum_{n=1}^{\infty}\frac{1}{n^x} < \infty$$

and for $x \geq 1 + \delta$ (with $\delta > 0$) the absolute convergence is uniform, so also the partial sums

$$\zeta_N(s) := \sum_{n=1}^{N}\frac{1}{n^s}$$

converge uniformly to $\zeta(s)$. Thus $\zeta(s)$ is analytic on the half plane $\mathrm{Re}(s) > 1$, by Morera's Theorem. Your favorite divergent series

$$\zeta(1) = \sum_{n=1}^{\infty}\frac{1}{n} = +\infty$$

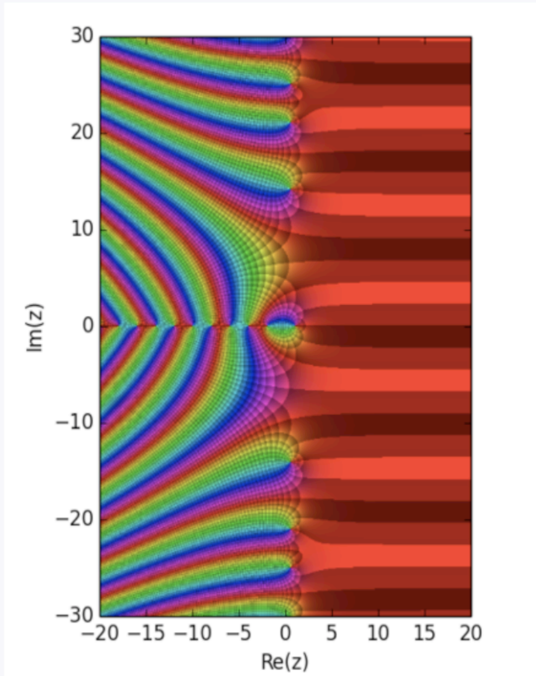
shows that $\zeta(s)$ is not analytic at $s = 1$. Somewhat surprisingly, $\zeta(s)$ can be extended to be analytic in all of $\mathbb{C} \setminus \{1\}$, however. (Such extensions must always be unique, it turns out.) The formulas for this extended function $\zeta(s)$ look different than the one that works on the half plane $\mathrm{Re}(s) > 1$.

The Riemann Zeta function has surprising connections to *number theory*, in particular to the *prime number theorem*, which is about how prime numbers are distributed in the natural numbers.
The *Riemann Hypothesis* is Riemann's conjecture from the 1800's, that all of the so-called non-trivial zeroes of the Riemann function lie on the line $\left\{\mathrm{Re}(s) = \frac{1}{2}\right\}$. (The other zeroes of the zeta function occur at the negative even integers.) It's considered one of the greatest unproven conjectures in mathematics, see for example the *Millenium prizes*. Of the billions of zeroes of the Riemann function which have been found, they're all on that line! Many results in number theory would follow if the Riemann hypothesis is true, so people are in the habit of proving theorems, where one of the assumptions is that the Riemann Hypothesis is true.

This is a great topic area for a research report in our course, if your interests go in this direction.

The output of the zeta function, plotted as a "graph" above the complex domain, with contours for the modulus and so that the color represents the argument of $\zeta(z)$. From wikipedia:

Riemann zeta function



The Riemann zeta function $\zeta(z)$ plotted with domain coloring.[1]

Basic features	
Domain	$\mathbb{C} \setminus \{1\}$
Codomain	$\mathbb{C}$

Specific values	
At zero	$-\frac{1}{2}$
Limit to +∞	1
Value at 2	$\frac{\pi^2}{6}$
Value at -1	$-\frac{1}{12}$
Value at -2	0